# Robust Estimation of Real Exchange Rate Process Half-life

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# Background

- The **real exchange rate** (in logs) is defined as

$$q_t \equiv s_t - \bar{p}_t^h + \bar{p}_t^f$$

with $s_t$ denoting the spot exchange rate, $\bar{p}_t^h$ the domestic price index and $\bar{p}_t^f$ the foreign price index.

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- The measure adopted in the literature to quantify the persistence of these deviations is the **half-life** of $\{q_t\}$ (Mark, 2001; Rossi, 2005) defined as the smallest $h$ such that

$$\psi(h) = \frac{1}{2} \mid \psi(0) = 1$$

with $\psi(t)$, $t \geq 0$ denoting the IRF of $\{q_t\}$.

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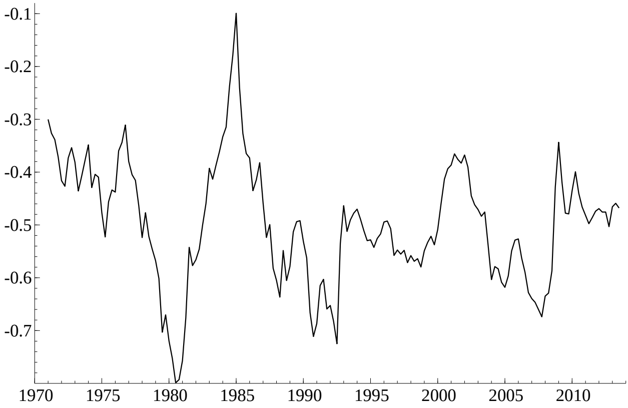
- In the present paper, we reconsider the extent of the PPP puzzle using **outliers robust inference** for ARMA processes.
- **Unaccounted outliers distort the half-life** estimates since they alter the autocorrelation structure of the observed time-series (Tsay, 1986) and hence the **IRF**.

# Motivation (cont'd)

Figure: USD/GBP Real Exchange Rate.



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- We consider a **Dummy Saturation** type procedure to detect and model the outliers observations; we check the procedure to deliver the correct retention rates of the dummies.
- We test the **PPP** for a group of countries by estimating the half-life of the real exchange rates with and without outliers detection.

# Real Exchange Rate Model

- Let the process followed by $\{q_t\}$ be described by

$$q_t = q_0 + \sum_{i=1}^{k} \delta_i V_i(L) \mathbf{1}(t = T_i) + v_t \quad (1)$$

$$\phi(L) v_t = \theta(L) \varepsilon_t \qquad t = 1, \ldots, T \quad (2)$$

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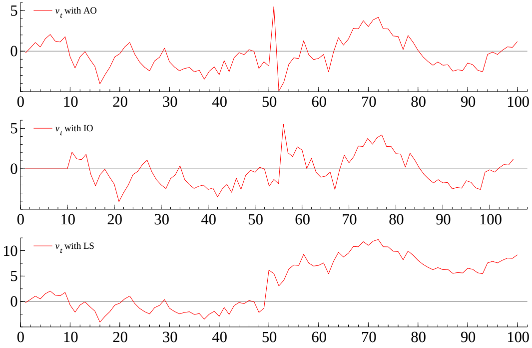
  - $\mathbf{1}(t = T_i)$ is an impulse indicator assuming value $1$ for $t = T_i$ and $0$ otherwise;
  - $\phi(L) = 1 - \phi_1 L - \cdots - \phi_p L^p$ and $\theta(L) = 1 - \theta_1 L - \cdots - \theta_q L^q$ are lag polynomials with roots outside the unit circle, and $\varepsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_\varepsilon^2)$.

# Real Exchange Rate Model (cont'd)

- Three specifications of $V_i(L)$ are particularly relevant to our analysis:

| | |
|---|---|
| $V_i(L) = 1$ | **Additive Outlier (AO)** |
| $V_i(L) = \phi^{-1}(L)\theta(L)$ | **Innovative Outlier (IO)** |
| $V_i(L) = (1 - L)^{-1}$ | **Level Shift (LS)** ($(1 - L)^{-1}\mathbf{1}(t = T_i) = \mathbf{1}(t \geq T_i)$). |



# Estimation

- It is convenient to rewrite (1)-(2) in matrix notation, to give

$$\mathbf{q} = \mathbf{W}\boldsymbol{\delta} + \mathbf{v} \qquad \mathbf{v} = \phi^{-1}(L)\theta(L)\varepsilon.$$

where $\mathbf{W}$ is a matrix of size $(1 + k^A + k^L + k^I) \times T$ mostly made up of $0$-$1$ entries.

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- Estimation of the unknowns (regression coefficients and time series parameters) can be obtained maximising the following likelihood

$$\ell(\boldsymbol{\delta}, \boldsymbol{\phi}, \boldsymbol{\theta}, \sigma_\varepsilon^2) = -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log\sigma_\varepsilon^2 - \frac{1}{2}\log|\boldsymbol{\Omega}| - \frac{1}{2\sigma_\varepsilon^2}(\mathbf{q} - \mathbf{W}\boldsymbol{\delta})^\top \boldsymbol{\Omega}^{-1}(\mathbf{q} - \mathbf{W}\boldsymbol{\delta}).$$

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- **Problem**: the matrix of regressors $\mathbf{W}$ is not known. This amounts to the problem of **selecting the outlying observations**.

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- We consider a **ML** procedure built around the **Dummy Saturation** principle (Hendry, 1999; Hendry et al., 2008; Johansen and Nielsen, 2009).

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**Steps**

1. **Initial ARMA order**: select using AIC the ARMA order under the null of no outliers, i.e. find the model

$$\phi(L)(q_t - q_0) = \theta(L)\varepsilon_t,$$

and denote the corresponding order with $(\tilde{p}, \tilde{q})$.

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   2.1 Saturate the model determined at Stage 1 with **AO**s.
      (a) Add the first half of AOs, say $x_{j,t}$, $j = 1, \ldots, \lfloor T/2 \rfloor$, and estimate by ML the following regression

$$q_t = q_0 + \sum_{j=1}^{\lfloor T/2 \rfloor} \delta_j^A x_{j,t} + \phi^{-1}(L)\theta(L)\varepsilon_t. \quad (3)$$

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   2.2 Repeat steps (a)-(d) for LSs and then IOs in order to get $\ddot{\mathbf{Y}}$ and $\ddot{\mathbf{Z}}$, the matrix containing the retained LSs and IOs respectively.

# Outliers Detection Approach (cont'd)

3. **Final model selection**: estimate by ML the following regression with ARMA errors

$$q_t = q_0 + \ddot{\mathbf{x}}_t^\top \boldsymbol{\delta}^A + \ddot{\mathbf{y}}_t^\top \boldsymbol{\delta}^L + \phi^{-1}(L)\theta(L)(\varepsilon_t + \ddot{\mathbf{z}}_t^\top \boldsymbol{\delta}^I) \tag{4}$$

and drop the not significant outliers. Estimation of (4) is iterated until the included outliers are all statistically significant and the ARMA order is modified accordingly.

# Outliers Detection Approach (cont'd)

- We check by simulation that the procedure delivers the **correct retention rates** of the dummies:

| $T = 100$ | | | $T = 200$ | | | $T = 300$ | | |
|---|---|---|---|---|---|---|---|---|
| AO | IO | LS | AO | IO | LS | AO | IO | LS |
| | | | | $\alpha = 0.01$ | | | | |
| | | | | $n = 2$ | | | | |
| 1.242 | 6.472 | 0.901 | 1.215 | 6.488 | 0.983 | 1.092 | 6.651 | 0.989 |
| | | | | $n = 5$ | | | | |
| 1.189 | 2.125 | 0.989 | 1.079 | 2.088 | 0.973 | 1.062 | 2.105 | 1.000 |
| | | | | $n = 10$ | | | | |
| 1.116 | 1.530 | 0.993 | 1.096 | 1.493 | 0.972 | 1.045 | 1.479 | 1.004 |
| | | | | $n = 20$ | | | | |
| 1.116 | 1.462 | 1.041 | 1.054 | 1.273 | 1.005 | 1.040 | 1.233 | 1.002 |
| | | | | $\alpha = 0.05$ | | | | |
| | | | | $n = 2$ | | | | |
| 5.260 | 15.880 | 4.655 | 5.223 | 16.085 | 4.818 | 5.103 | 16.265 | 4.899 |
| | | | | $n = 5$ | | | | |
| 5.245 | 7.756 | 4.567 | 5.096 | 7.822 | 4.951 | 5.092 | 7.875 | 4.981 |
| | | | | $n = 10$ | | | | |
| 5.206 | 6.346 | 4.894 | 5.188 | 6.235 | 4.965 | 5.038 | 6.271 | 5.010 |
| | | | | $n = 20$ | | | | |
| 5.332 | 6.140 | 4.995 | 5.096 | 5.680 | 4.994 | 5.132 | 5.669 | 5.022 |

# Robust Half-life Computation

- Robust half-life estimates are obtained from the following **"cleaned" series**

$$\tilde{q}_t \equiv q_t - q_0 - \mathbf{x}_t^\top \boldsymbol{\delta}^A - \mathbf{y}_t^\top \boldsymbol{\delta}^L - \phi(L)^{-1}\theta(L)\mathbf{z}_t^\top \boldsymbol{\delta}^I = \phi^{-1}(L)\theta(L)\varepsilon_t.$$

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- Let $\psi(L) = \phi^{-1}(L)\theta(L)$ to give $\tilde{q}_t = \sum_{j=0}^{+\infty} \psi_j \varepsilon_{t-j}$, with $\sum_{j=0}^{+\infty} \psi_j^2 < \infty$ (under the assumption that the roots of $\phi(L)$ all lie outside the unit circle), such that $\lim_{j\to\infty} \psi_j = 0$.

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- Let $\psi(j) = \psi_j$, $j = 1, 2, \ldots, T$ and $\psi(0) = 1$ be the IRF, we are interested in finding the **first instant $h$ such that** $\psi(h) = 0.5$.

# Robust Half-life Computation (cont'd)

- Assuming $\{\tilde{q}_t\}_{t=1}^{T} \sim ARMA(p,q)$,

$$\underbrace{\begin{bmatrix} \tilde{q}_t \\ \tilde{q}_{t-1} \\ \vdots \\ \tilde{q}_{t-p+1} \\ \varepsilon_t \\ \varepsilon_{t-1} \\ \vdots \\ \varepsilon_{t-q+1} \end{bmatrix}}_{\boldsymbol{\xi}_t} = \underbrace{\begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_p & \theta_1 & \dots & \theta_q \\ 1 & 0 & \dots & & & & \\ & \ddots & & & & & \\ 0 & & 0 & \dots & & & \\ 0 & \dots & 1 & 0 & \dots & & \\ 0 & \dots & & & & & \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots \\ & & & & & \ddots & \\ 0 & 0 & \dots & 0 & 0 & & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 1 \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} \tilde{q}_{t-1} \\ \tilde{q}_{t-2} \\ \vdots \\ \tilde{q}_{t-p} \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \\ \vdots \\ \varepsilon_{t-q} \end{bmatrix}}_{\boldsymbol{\xi}_{t-1}} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\mathbf{G}} \varepsilon_t$$

The IRF can be obtained as

$$\psi(j) = \mathbf{e}(\mathbf{F}^j \mathbf{G})$$

where $\mathbf{e} = [1\ 0\ \dots\ 0]^\top$ is a selection vector.

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- To find the smallest value $h$ such that $\mathbf{e}(\mathbf{F}^h\mathbf{G}) = 0.5$ we use **interpolating splines**.

# Empirical Application: Data

- We analyse **US dollar bilateral exchange rates** for a group of developed countries: United Kingdom, Germany, France, Italy, Switzerland, Japan, South Africa, Mexico and the Euro Area (EMU).

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- Real exchange rate for the $i^{\text{th}}$ country ($q_{i,t}$) is computed from the **nominal exchange rate** ($s_{i,t}$, currency units for $1) and the **CPIs** ($p_{i,t}$ and $p_{US,t}$) as

$$q_{i,t} = s_{i,t} - p_{i,t} + p_{US,t}$$

# Empirical Application: Data

- We analyse **US dollar bilateral exchange rates** for a group of developed countries: United Kingdom, Germany, France, Italy, Switzerland, Japan, South Africa, Mexico and the Euro Area (EMU).
- Real exchange rate for the $i^{\text{th}}$ country ($q_{i,t}$) is computed from the **nominal exchange rate** ($s_{i,t}$, currency units for \$1) and the **CPIs** ($p_{i,t}$ and $p_{US,t}$) as

$$q_{i,t} = s_{i,t} - p_{i,t} + p_{US,t}$$

- Data are obtained from the FRED database at a **quarterly frequency** over the period 1971:1-2013:3 (max 171 obs, min 59 obs).

# Empirical Application: Half-life Computation Without Outliers Detection

| | $\hat{h}$ | $\hat{c}_{low}$ | $\hat{c}_{upp}$ | $(p, q)$ | AIC | J-B |
|---|---|---|---|---|---|---|
| UK | 1.78 | 0.85 | 2.26 | 4,3 | -584.61 | [0.0050]** |
| Germany | 1.86 | 1.11 | 4.82 | 2,2 | -400.49 | [0.3885] |
| France | 1.82 | 1.14 | 4.27 | 2,2 | -419.27 | [0.0456]* |
| Italy | 1.87 | 1.13 | 4.74 | 2,2 | -414.70 | [0.0306]* |
| Switzerland | 7.05 | 1.89 | 12.12 | 1,1 | -541.48 | [0.9039] |
| South Africa | 5.27 | 1.40 | 7.62 | 2,1 | -471.59 | [0.0001]** |
| Japan | 6.60 | 2.31 | 11.02 | 5,1 | -545.81 | [0.0503] |
| Mexico | 0.99 | 0.31 | 1.13 | 3,3 | -211.69 | [0.0000]** |
| Euro Area | 3.30 | 0.67 | 4.38 | 1,1 | -208.00 | [0.2370] |

**Notes**: $\hat{h}$ denotes the annualised half-life estimate, $\hat{c}_{low}$ and $\hat{c}_{upp}$ are the lower and upper endpoint of the bootstrapped confidence interval, $(p, q)$ denotes the ARMA order, $AIC$ the Akaike Information Criterion and J-B the $p$-value of the Jarque-Bera test with '**' and '*' denoting rejection of the null of Normality at 1% and 5% significance level respectively.

# Empirical Application: Half-life Computation With Outliers Detection

| | $\hat{h}$ | $\hat{c}_{low}$ | $\hat{c}_{upp}$ | $(p, q)$ | AOs | IOs | LSs | AIC | J-B |
|---|---|---|---|---|---|---|---|---|---|
| UK | 1.25 | 0.86 | 1.93 | 4,3 | 1981(3)<br>1985(1)<br>1988(2) | 2008(4) | 1990(3)<br>1992(4) | -658.67 | [0.4640] |
| Germany | 1.89 | 1.27 | 3.72 | 2,2 | | | 1974(3)<br>1975(3)<br>1984(3)<br>1988(3) | -412.26 | [0.7323] |
| France | 1.85 | 1.24 | 3.88 | 2,2 | 1985(1) | 1991(2) | | -425.33 | [0.1589] |
| Italy | 1.66 | 1.03 | 4.45 | 2,2 | 2000(4) | 1976(1)<br>1984(3)<br>1992(3) | | -425.99 | [0.0117]* |
| Switzerland | 2.85 | 1.15 | 4.82 | 1,1 | 1985(1) | 1971(1)<br>1978(1) | | -558.30 | [0.7162] |
| South Africa | 1.95 | 0.96 | 2.42 | 3,2 | | 2001(4)<br>2008(4) | 1975(4)<br>1998(3) | -516.03 | [0.0000]** |
| Japan | 3.59 | 1.85 | 5.07 | 5,1 | 1979(4)<br>1995(2) | 1971(1)<br>1998(4)<br>2008(4) | 1978(3)<br>2013(1) | -593.94 | [0.8416] |
| Mexico | 2.05 | 0.38 | 3.26 | 2,3 | | 1995(1)<br>2008(4) | 1995(2) | -306.87 | [0.3196] |
| Euro Area | 1.31 | 0.51 | 2.19 | 1,1 | 2000(4) | | 2003(4)<br>2004(1) | -218.31 | [0.2711] |

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- For “problematic” countries, **modelling outliers** seems to drastically **reduce** the extent of **the PPP puzzle**.
- For countries where the puzzle is less evident or absent, half-life estimates are not affected.
- Benefits of accounting for outliers are in any case evident in **tighter confidence intervals** and **restored normality**.
- **Four** outliers retained on average. The most recurring is the IO in the fourth quarter of 2008.

# Testing for Non-linear Effects

- Part of the literature models $\{q_t\}$ using **non-linear models** (TAR, SETAR, ...) following the "bands of inaction" argument raised by Taylor (2001).

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- We employ the **BDS statistic** (Brock et al., 1996) which is based on the concept of **correlation integral** and aims at measuring the frequency with which temporal patterns repeat over time.

# Testing for Non-linear Effects (cont'd)

| $\eta = 0.5$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| UK | 0.003** | 0.005** | 0.002** | 0.000** |
| | 0.484 | 0.313 | 0.313 | 0.004** |
| Germany | 0.116 | 0.025* | 0.960 | 0.097 |
| | 0.002** | 0.285 | 0.689 | 0.136 |
| France | 0.447 | 0.992 | 0.603 | 0.711 |
| | 0.116 | 0.313 | 0.741 | 0.857 |
| Italy | 0.001** | 0.000** | 0.000** | 0.000** |
| | 0.037* | 0.006** | 0.007** | 0.000** |
| Switzerland | 0.749 | 0.535 | 0.126 | 0.022* |
| | 0.772 | 0.294 | 0.562 | 0.352 |
| South Africa | 0.003** | 0.001** | 0.000** | 0.000** |
| | 0.001** | 0.000** | 0.000** | 0.000** |
| Japan | 0.689 | 0.298 | 0.230 | 0.478 |
| | 0.757 | 0.407 | 0.711 | 0.332 |
| Mexico | 0.000** | 0.002** | 0.052 | 0.099 |
| | 0.000** | 0.004** | 0.067 | 0.072 |
| EMU | 0.215 | 0.002** | 0.000** | 0.000** |
| | 0.555 | 0.522 | 0.119 | 0.555 |

**Notes**: ** and * denote presence of non-linear effects at 1% and 5% significance level.

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  2. In any case, robust estimation allows to obtain **tighter CIs** and **to restore normality**.
  3. **Presence of non-linear effects** is mixed without modelling outliers and it becomes **even less evident when accounting for outliers**.

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